Homework #1: Dynamic Programming
ECON 6313
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1. Each period, a household seeks to maximize its expected lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to its asset accumulation equation and its initial level of assets, A_0:

$$A_{t+1} = R(A_t - C_t),$$

$$A_0 = \overline{A}_0,$$

where C_t is consumption, A_t is assets, R is the constant gross nominal interest rate, $0 \leq \beta < 1$ is the discount factor, and $\sigma > 0$ is a preference parameter. In your answer, assume that

$$R^{1-\sigma} < 1/\beta,$$

and

$$\lim_{T \longrightarrow \infty} \left(\frac{1}{R}\right)^T A_{t+T} = 0.$$

(a) State the household's problem using the Bellman equation. Next, form the Lagrangian for the household's problem via the Bellman equation, where λ_t is the Lagrange multiplier on the household's asset accumulation equation. Lastly, find the first-order conditions with respect to C_t, A_{t+1}, and λ_t.

Answer

The Bellman equation is

$$V(A_t) = \frac{C_t^{1-\sigma}}{1-\sigma} + \beta V(A_{t+1}),$$

subject to

$$A_{t+1} = R(A_t - C_t).$$

The Lagrangian is

$$\pounds = \frac{C_t^{1-\sigma}}{1-\sigma} + \beta V(A_{t+1}) + \lambda_t[R(A_t - C_t) - A_{t+1}].$$

The first-order conditions with respect to C_t, A_{t+1}, and λ_t are

$$C_t^{-\sigma} = \lambda_t R,$$

$$\beta V'(A_{t+1}) = \lambda_t,$$

$$A_{t+1} = R(A_t - C_t).$$

(b) Derive the Benveniste-Scheinkman condition. In your answer, specify the values of $V'(A_t)$ and $V'(A_{t+1})$.

Answer

Substitute the asset constraint equation into the Bellman equation:

$$V(A_t) = \frac{C_t^{1-\sigma}}{1-\sigma} + \beta V(R(A_t - C_t)).$$

Next, totally differentiate the resulting equation:

$$V'(A_t) = C_t^{-\sigma} + \beta V'(A_{t+1})R - \beta V'(A_{t+1})R.$$

Since $C_t^{-\sigma} = \beta R V'(A_{t+1})$, the Benveniste-Scheinkman condition becomes

$$V'(A_t) = +\beta R V'(A_{t+1}).$$

Since $V'(A_t) = C_t^{-\sigma}/(\beta R)$ and $V'(A_{t+1}) = C_{t+1}^{-\sigma}/(\beta R)$, the Benveniste-Scheinkman condition becomes

$$C_t^{-\sigma} = \beta R C_{t+1}^{-\sigma},$$

$$C_{t+1} = (\beta R)^{1/\sigma} C_t.$$

(c) Show the optimal policy functions for C_t and A_{t+1} in the form $C_t = (1-\gamma_C)A_t$ and $A_{t+1} = \gamma_A A_t$, respectively? In your answer make sure to define γ_C and γ_A.

Answer

To solve for the optimal policy functions, forward solve for A_t using the asset accumulation equation

$$A_{t+1} = R(A_t - C_t),$$

$$A_t = \frac{A_{t+1}}{R} + C_t,$$

$$A_t = \frac{A_{t+2}}{R^2} + C_t + \frac{C_{t+1}}{R},$$

$$A_t = \left(\frac{1}{R}\right)^T A_{t+T} + \sum_{j=0}^{T-1} \left(\frac{1}{R}\right)^j C_{t+j}.$$

Since $\lim_{T \longrightarrow \infty} \left(\frac{1}{R}\right)^T A_{t+T} = 0$,

$$A_t = \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j C_{t+j}.$$

Using the Benveniste-Scheinkman condition, $C_{t+1} = (\beta R)^{1/\sigma} C_t$, the value of A_t becomes

$$A_t = \sum_{j=0}^{\infty} \left(\frac{(\beta R)^{1/\sigma}}{R}\right)^j C_t,$$

$$A_t = \sum_{j=0}^{\infty} \left(\beta^{1/\sigma} R^{(1/\sigma)-1}\right)^j C_t.$$

Since $\sum_{j=0}^{\infty} \left(\beta^{1/\sigma} R^{(1/\sigma)-1}\right)^j = 1/(1 - \beta^{1/\sigma} R^{(1/\sigma)-1})$, the value of A_t becomes

$$A_t = \frac{1}{1 - \beta^{1/\sigma} R^{(1/\sigma)-1}} C_t.$$

The value of C_t is then can be calculated as a function of A_t

$$C_t = [1 - \beta^{1/\sigma} R^{(1/\sigma)-1}] A_t,$$

where

$$\gamma_C = \beta^{1/\sigma} R^{(1/\sigma)-1}.$$

Using the asset accumulation equation, $A_{t+1} = R(A_t - C_t)$, we get

$$A_{t+1} = R(A_t - C_t),$$

$$A_{t+1} = R(A_t - (1 - \beta^{1/\sigma} R^{(1/\sigma)-1}) A_t),$$

$$A_{t+1} = R(\beta^{1/\sigma} R^{(1/\sigma)-1}) A_t,$$

$$A_{t+1} = \beta^{1/\sigma} R^{1/\sigma} A_t,$$

where

$$\gamma_A = \beta^{1/\sigma} R^{(1/\sigma)}.$$

2. Consider a Ramsey model where a social planner seeks to maximize utility:

$$\sum_{t=0}^{\infty} \beta^t \ln (C_t)$$

subject to the economy-wide budget constraint:

$$K_{t+1} = K_t^{\alpha} - C_t,$$

where C_t is consumption, K_t is capital, $0 \leq \beta < 1$ is the discount factor, and $0 < \alpha < 1$. Now, suppose we guess the value function has the following form:

$$V^g(K_t) = E + F \ln(K_t),$$

where E and F are the parameters that need to be derived.

(a) Using the guess for the value function, state the household's problem using the Bellman equation. Next, form the Lagrangian for the social planner's problem via the Bellman equation, where λ_t is the Lagrange multiplier on the economy-wide budget constraint. Lastly, find the first-order conditions with respect to C_t, K_{t+1}, and λ_t.

Answer

The Bellman equation for the social planner's problem is

$$V(K_t) = \ln (C_t) + \beta[E + F \ln(K_{t+1})],$$

subject to

$$K_t^\alpha = C_t + K_{t+1}.$$

The Lagrangian is

$$\pounds = \ln(C_t) + \beta[E + F\ln(K_{t+1})] + \lambda_t[K_t^\alpha - C_t - K_{t+1}].$$

The first-order conditions with respect to C_t, K_{t+1}, and λ_t are

$$1/C_t = \lambda_t,$$

$$\frac{\beta F}{K_{t+1}} = \lambda_t,$$

$$K_t^\alpha = C_t + K_{t+1}.$$

(b) Find the policy functions for C_t and K_{t+1} as a function of K_{t+1} given the guess for the value function.

Answer

Combine the first-order conditions for C_t and K_{t+1}

$$\frac{1}{C_t} = \frac{\beta F}{K_{t+1}},$$

$$K_{t+1} = \beta F C_t.$$

Substitute this equation into the budget constraint to find the policy function for C_t

$$K_t^\alpha = C_t + \beta F C_t,$$

$$C_t = \frac{K_t^\alpha}{1+\beta F}.$$

Substituting the policy function for C_t into the budget constraint to generate the policy function for K_{t+1}:

$$K_t^\alpha = \frac{K_t^\alpha}{1+\beta F} + K_{t+1},$$

$$K_{t+1} = \frac{BF}{1+\beta F} K_t^\alpha.$$

(c) Compare the new value function with the initial guess and then solve for values of E and F. Using the values for E and F, solve the policy functions for C_t and K_{t+1}.

Answer

Substitute the policy functions for C_t and K_{t+1} into the Bellman equation assuming that $V^g(K_t) = E + F\ln(K_t)$

$$V(K_t) = \ln\left(\frac{K_t^\alpha}{1+\beta F}\right) + \beta\left[E + F\ln\left(\frac{BF}{1+\beta F}K_t^\alpha\right)\right],$$

$$V(K_t) = \beta E + \ln\left(\frac{1}{1+\beta F}\right) + \beta F\ln\left(\frac{\beta F}{1+\beta F}\right) + \alpha(1+\beta F)\ln(K_t).$$

Thus, F and E equal

$$F = \alpha(1 + \beta F),$$

$$F = \frac{\alpha}{1 - \alpha\beta},$$

and

$$E = \beta E + \ln\left(\frac{1}{1 + \beta F}\right) + \beta F \ln\left(\frac{\beta F}{1 + \beta F}\right),$$

$$E(1 - \beta) = \ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta}\ln(\alpha\beta),$$

$$E = \frac{\ln(1 - \alpha\beta)}{1 - \beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1 - \alpha\beta)(1 - \beta)}.$$

Using the values for E and F, we can derive the policy functions for C_t and K_{t+1}

$$C_t = \frac{K_t^\alpha}{1 + \frac{\beta\alpha}{1 - \alpha\beta}},$$

$$C_t = (1 - \alpha\beta)K_t^\alpha.$$

$$K_{t+1} = \frac{\frac{\alpha B}{1 - \alpha\beta}}{1 + \frac{\alpha\beta}{1 - \alpha\beta}} K_t^\alpha,$$

$$K_{t+1} = \alpha B K_t^\alpha.$$