Homework #2: Linear Stochastic Systems
ECON 6313
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1. Consider the following autoregressive moving-average (ARMA) model:

$$m_t = 0.51 + 0.5m_{t-1} + \eta_t - 0.9\eta_{t-1} - 0.1\eta_{t-2}.$$

where η_t is an independently and identically distributed random variable that has a zero mean. Show how this ARMA(1,2) model can be incorporated into the state space system:

$$(Y_t - \overline{Y}) = \Pi(S_t - \overline{S})$$
$$(S_t - \overline{S}) = M(S_{t-1} - \overline{S}) + \epsilon_t$$

Be specific on the elements in the vectors $(Y_t - \overline{Y})$, $(S_t - \overline{S})$, and ϵ_t and the matrices Π and M.

2. Consider the following expectational Phillips curve equation:

$$\pi_t = \beta E_t \pi_{t+1} + \varphi(y_t - \overline{y})$$

where $0 < \beta < 1$. Suppose that $x_t = (y_t - \overline{y})$ evolves according to a first order difference equation, $x_t = \rho x_{t-1} + \varepsilon_t$, where $0 < \rho < 1$. Solve this expression for a solution of the form $\pi_t = \theta x_t$ and show how θ depends on β, ρ, and φ.

3. Consider the following state space system

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} 0.8 & -0.1 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} s_{1,t-1} \\ s_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \tag{2}$$

where (1) is the prediction equation, (2) is the updating equation and $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independently and identically distributed random variables that have a zero mean.

(a) What is the forecast of $y_{1,t+2}$ and $y_{2,t+2}$ at date t?

(b) Find the relevant eigenvalues and eigenvectors for the matrix M in the updating equation. Indicate which eigenvector corresponds with each eigenvalue.

(c) Calculate the following Kronecker product for the matrix M: $Kron(M, M)$

(d) Calculate the unconditional variance for the state vector, V_{SS}, in above state space model given that $var(\varepsilon_{1,t}) = var(\varepsilon_{2,t}) = 1$, $cov(\varepsilon_{1,t}, \varepsilon_{2,t}) = 0$, and

$$[I_4 - Kron(M, M)]^{-1} = \begin{bmatrix} 2.5091 & -0.3117 & -0.3117 & 0.0750 \\ 0.6233 & 1.5741 & -0.1500 & -0.0866 \\ 0.6233 & -0.1500 & 1.5741 & -0.0866 \\ 0.3000 & 0.1733 & 0.1733 & 1.3142 \end{bmatrix}$$

(e) Calculate $cov(y_{1,t}, y_{2,t-1})$. (Hint: You need to calculate $\Gamma_{YY}(1)$.)