Homework #6: Solving Rational Expectations Models

ECON 5163

Benjamin Keen

Let us consider a neoclassical growth model where there is a stochastic level of total factor productivity. At any date t, households in the economy have preferences over the amounts of consumption, C_t, and leisure, L_t, which they receive in the current period and all other periods. These preferences are represented by following the utility function:

$$U = \sum_{i=0}^{\infty} \beta^i E_t \left[\ln(C_{t+i}) + \theta \ln(L_{t+i})\right] \tag{1}$$

where β is a discount factor ($0 < \beta < 1$). There are four equations which describe aspects of the constraints on the welfare of these households. First, their available time, which is normalized to one, must be allocated between leisure and labor, N_t

$$L_t + N_t = 1. \tag{2}$$

Second, the available output, Y_t, must be allocated between consumption and investment, I_t, uses

$$C_t + I_t = Y_t \tag{3}$$

Third, a Cobb-Douglas production function dictates how much output will be produced from inputs of labor and capital, K_t,

$$Z_t(K_t)^{\alpha}(N_t)^{1-\alpha} = Y_t \tag{4}$$

where $Z_t > 0$ is the time varying level of productivity that is exogenous to the model. Fourth, the capital stock evolves according to

$$K_{t+1} = I_t + (1-\delta)K_t \tag{5}$$

where δ is the capital depreciation rate. Finally it is assumed that K_t is predetermined by investment decisions made in period $t-1$ or earlier.

1. We need to simplify the model a bit. First, substitute (2) into the utility function, (1), in order to eliminate L_t from the system of equations. Second, combine equations (3)-(5) into a single constraint so that I_t and Y_t "drop out" of the system.

2. The social planner maximizes U in (1) subject to the economy resource constraint derived in problem #1 by choosing $\{C_{t+i}\}_{i=0}^{\infty}$, $\{N_{t+i}\}_{i=0}^{\infty}$ and $\{K_{t+1+1}\}_{i=0}^{\infty}$ and by taking K_t as given. Form a Lagrangian which attaches a multiplier λ_t to the economy resource constraint and then find the first-order conditions for this problem with respect to C_t, K_{t+1}, N_t, and λ_t.

3. One additional condition imposed on this model is

   $$\lim_{i\to\infty} \beta^i \lambda_{t+i} K_{t+1+i} = 0.$$

   What is the name of this condition? Why is this condition a necessary part of our dynamic analysis? What is the analogous condition in a finite horizon model of capital accumulation in which the economy can end up at the end of the last period, T, with no less than $K_T = \overline{K} > 0$.

4. Suppose you were told that $Z_t = Z^*$ and $N_t = N^*$ in the steady state and told to drop the first-order condition for N_t from the list of equations which describe the steady state. Given this information, what are the steady state values of C_t, K_t, and λ_t?

5. Linearize the four first-order conditions around their steady state and express the results in terms of the percent deviation of the endogenous and exogenous variables from their steady state. Next, place the four linearized equations into the following matrix form:
$$AE_t\widehat{Y}_{t+1} = B\widehat{Y}_t + C_0\widehat{X}_t + C_1E_t\widehat{X}_{t+1},$$
where $\widehat{Y}_t$ is the percent deviation of the endogenous variables from their steady state and $\widehat{X}_t$ is the percent deviation of the exogenous variable, Z_t, from its steady state. Is the matrix A singular or nonsingular?

6. If the matrix A is singular, eliminate $\widehat{C}_t$ from the system of equations by substituting out the first-order condition for C_t. In this case, is the matrix A nonsingular? If not, repeat the same procedure except eliminate $\widehat{N}_t$ from the system of equations by substituting out the first-order condition for N_t. In this case, is the matrix A nonsingular?